A Smoother Pebble

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Mathematical Explorations

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Introduction

I do not know what I may appear to the world; but to myself, I seem to have been only like a child playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

—ISAAC NEWTON

THIS BOOK EXPLORES PATHS on the mathematical seashore. Paths are the accumulated footprints of those who came before. There are many paths to choose from — some leading to minor curiosities and others leading to important mathematical goals. In this book, I intend to point out a few paths that I believe are both curious and important — paths with mainstream destinations.

I intend to show mathematics as a human endeavor, not a cold unapproachable monolithic perfection. The search for a useful, convincing, and reliable understanding of number and space has had many successes, but also a few false starts and wrong turns. Botticelli's famous painting *The Birth of Venus* shows Venus born of the foam of the sea, not as an infant but as a beautiful woman, divine in every detail. Mathematics, on the other hand, did not achieve such instant perfection at birth. Her growth has been long and tortuous, and perfection may be out of reach.

Part I of this book deals with the concept of number. We begin with the curious method of the ancient Egyptians for representing fractions. Fractions were a difficult concept for the ancients, and still are for today's schoolchildren. Unguided, the mathematical pioneers discovered over centuries what today's schoolchildren, guided by their teachers, learn in weeks. The Egyptian method seems clumsy to us, yet we will see that it provides some advantages in dividing five pies among seven people.

Part II is devoted to geometry. We will visit a fantastic universe called Tubeland. The efforts of Tubelanders to understand their world is a reflection of the efforts of our scientists to understand ours. Question: What geometric device was unknown in 1800, a promising innovation in 1900, and a universal commonplace in mathematics, science, business, and everyday life in 2000? Answer: *Graphs*.

Part III is concerned with algebra, the language of mathematics. Solving equations was a passionate undertaking for five Italian mathematicians of the sixteenth century. For them, algebraic knowledge was booty of great value, the object of quarrels, conspiracies, insults, and fiery boasts. Later, we discover what a catalog of wallpaper ornaments has to do with algebra.

Part IV introduces the smoother pebble discovered by Newton and Leibniz: the calculus. The basic concepts are introduced by means of a six-minute automobile ride. Later, we witness the competition for the fastest roller coaster.

I hope that the reader gains from this book new meaning and new pleasure in mathematics.

Part I

Bridging the Gap

Science is the attempt to make the chaotic diversity of our sense-experiments correspond to a logically uniform system of thought.

—Albert Einstein (1879–1955)

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Ancient Fractions

The Eye of Horus burning with fire before my eyes.

—THE BOOK OF THE DEAD, 1240 BCE
(translated by E.A. Wallis Budge)

POSITIVE WHOLE NUMBERS—the *natural numbers*—fill the fundamental human need for counting, but, additionally, a civilized society requires fractional numbers for the orderly division of land and goods—*artificial numbers* that fill in the gaps between the natural numbers.

Getting fractions right is the first slippery step in the mathematical education of many schoolchildren, a place where many fall. So it was also in the history of mathematics. The ancient Egyptians took a wrong turn. Only after thousands of years did others find the right path. This detour is now all but forgotten, and there is no danger that we will repeat this mistake. Since fractions were not easy for the Egyptians, we can be more understanding of the difficulties that our schoolchildren experience. Furthermore, Egyptian fractions are a source of curious problems, interesting in their own right.

The ancient Babylonians must be given high marks for their treatment of fractions. Babylonian fractions were quite similar to today's decimal fractions; however, the Babylonian system was based on the number 60 instead of 10. We still use Babylonian fractions when we use minutes and seconds to measure time and angles.

The German mathematician Leopold Kronecker (1823–91) said, "God created the whole numbers. All the rest is the work of man." There is essentially one way to understand the natural numbers. However, there are several different ways to define fractional numbers—also known as *rational numbers*. The fractions in current use—a numerator and denominator separated by a bar, for example, 5/7—we call *common fractions*. This notation originated in India in the twelfth century and soon spread to Europe, but the underlying concept—*ratios of commensurable magnitudes*—is from the ancient Greeks. However, common fractions are not the only way to conceive of fractions. In this chapter, we will see that the ancient Egyptians and Babylonians had different methods. In Chapter 2, we will see yet another method of defining fractional numbers.

The Egyptian Unit Fractions

The Rhind Papyrus, a scroll that measures 18 feet by 13 inches, is the most important source of information concerning ancient Egyptian mathematics. It was found in Thebes and purchased by Scottish Egyptologist Henry Rhind in 1858; it has been held by the British Museum since 1863. The scroll was written by the scribe Ahmes (1680?–1620? BCE), who states that he copied the material from older sources—1850 BCE or earlier.

The scroll—consisting of two tables and 87 problems—is a textbook of ancient Egyptian mathematics. Some of the problems deal with areas and volumes; however, a considerable part of the scroll is concerned with the ancient Egyptian arithmetic of fractions. Despite obvious shortcomings, these curious methods persisted for thousands of years. In fact, we will see that Leonardo of Pisa (1175?–1230?)[1] made an important contribution to the theory of Egyptian *unit fractions*.

The ancient Egyptians devised a concept of fractions that seems strange—even bizarre—to us today. A fraction with numerator equal to 1 (e.g., $1/3$, $1/7$) is called a unit fraction. The Egyptians denoted unit fractions by placing the eye-shaped symbol ⬭ ("the eye of Horus") above a natural number to indicate its *reciprocal*. We approximate this notation by using, for example, $\overline{7}$ to represent $1/7$.

The Egyptians had a special notation for $2/3$, but all other fractions were represented as sums of distinct unit fractions. For example, for $5/7$ they could have written

$$\frac{5}{7} = \overline{2} + \overline{7} + \overline{14} \tag{1.1}$$

We confirm this by the following computation:

$$\frac{1}{2} + \frac{1}{7} + \frac{1}{14} = \frac{7+2+1}{14} = \frac{10}{14} = \frac{5}{7} \tag{1.2}$$

Similar computations show that the fraction $5/7$ can also be represented as

$$\frac{5}{7} = \overline{2} + \overline{5} + \overline{70} \tag{1.3}$$

or

$$\frac{5}{7} = \overline{3} + \overline{4} + \overline{8} + \overline{168} \tag{1.4}$$

It did not occur to the Egyptians to use *two* numbers, a numerator and denominator, to represent a fraction. When we write 5/7, and when we calculate as in equation (1.2) above, we are departing from the ancient Egyptian mode of thought.

Why did the Egyptians avoid repetitions of unit fractions? Why did they feel, for example, that $\overline{7}+\overline{7}+\overline{7}+\overline{7}+\overline{7}$ is unacceptable? One can only speculate, but perhaps they felt that it is not permissible to express a fraction as a sum of five unit fractions when three (as in equations (1.1) and (1.3)) are all that are needed.

The fact—show in equations (1.1), (1.3), and (1.4)—that 5/7 has more than one representation as a sum of unit fractions indicates a serious flaw in the Egyptian system for fractions. How is it possible that such an awkward system remained in use for thousands of years? There are several possible answers:

1. The system was adequate for simple needs.
2. The system was sanctioned by tradition.
3. The scribes who used the system had no wish to diminish their reputations for wizardry by simplifying the system.
4. It really does take thousands of years to get the bright idea that one can use *two* natural numbers—numerator and denominator—to specify a fraction.

Aside from the merit of the above speculations, there are certain real advantages in using Egyptian unit fractions for problems involving the division of goods. A fair method of division divides the whole into a number of pieces and specifies the pieces in each share. If we assume that the goods in question are *fungible*,[2] then the most important requirement for a method of fair division is that the total size of each share be identical regardless of the number and shape of the pieces forming each share. However, there are other considerations. For example, it adds to the *appearance* of fairness if the shares are identical—not only in aggregate size, but also in the number and shape of the pieces. Furthermore, the number of pieces should not be excessive. As shown by the following example, unit fractions can lead to a division of goods with certain advantages.

Example 1.1. Divide 5 pies among 7 people, Ada, Ben, Cal, Dot, Eli, Fay, and Gil, (a) using ordinary arithmetic, (b) using Egyptian unit fractions.

(a) **Two methods using ordinary arithmetic.**

Method 1:

1. Ada gets 5/7 of the first pie.
2. Ben gets 2/7 of the first pie and 3/7 of the second pie.
3. Cal gets 4/7 of the second pie and 1/7 of the third pie.
4. Dot gets 5/7 of the third pie.
5. Eli gets 1/7 of the third pie and 4/7 of the fourth pie.

6. Fay gets $3/7$ of the fourth pie and $2/7$ of the fifth pie.
7. Gil gets $5/7$ of the fifth pie.

Objection 1: Disagreements can arise because the shares contain different sized pieces.

Method 2: Divide each of the five pies into seven equal pieces. A share consist of five of these pieces.

Objection 2: Too many pieces in each share.

(b) **A method using Egyptian unit fractions.** In this method each share consists of just three pieces, and all the shares have the same appearance. Since, according to equation (1.1), $5/7 = 1/2 + 1/7 + 1/14$, we proceed as follows:

1. Give everyone half of a pie. This leaves a pie and a half to be distributed.
2. Cut the remaining whole pie in sevenths. Give each of the seven people one of these pieces. There remains half a pie.
3. Cut the half pie in seven equal pieces. Give each person one of these pieces.

The Egyptian method avoids the worst features of each of the two modern methods. It beats Method 1 on the grounds of Objection 1 and Method 2 on the grounds of Objection 2.

Egyptian arithmetic

Like schoolchildren of today, the Egyptians needed basic arithmetic as a background for computing with fractions. The Rhind Papyrus gives examples illustrating a complex collection of arithmetic techniques. We will consider some of the methods of multiplication and division—especially as they relate to fractions.[3]

Multiplication

The ancient Egyptians did not use our familiar methods of multiplication and division. The basic method of multiplication, which proceeds by successive doubling, is illustrated in Table 1.1(a). Successive doubling involves exactly the same arithmetic as *Russian peasant multiplication* (see Table 1.1(b)). Both methods convert one of the factors to the *binary system,* the arithmetic basis for the modern digital computer. In multiplying 13×14, the factor 13 is converted to binary ($13 = 8 + 4 + 1 = 1101_2$)—in Table 1.1(a) by starring certain items and in Table 1.1(b) by striking out certain items. The Russian peasant method is an improvement because it gives a mechanical process—an algorithm—for the binary conversion.