

Lecture Notes in Physics

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A Renormalization Group Analysis
of the Hierarchical Model
in Statistical Mechanics



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INTRODUCTION

The so-called renormalization group (RG) theory which has seen a vigorous development in the past few years has considerably strengthened our understanding of phenomena near to phase transitions of statistical mechanics, and it has also given some insight into the difficulties of relativistic quantum field theories. Maybe the main virtue of the RG theory has been to ask the right questions, namely to put the study of collective phenomena (that is the cooperative behaviour of many particles or modes) into a good perspective. The method consists of studying the behaviour of a physical system under a change of scale. The study of this question can be separated into two parts :

Firstly, to ask in which way the microscopic physical laws transform under such a change of scale, and secondly, to ask why and how information about the system near a "critical" situation can be obtained once the transformations of these microscopic laws are known.

The second question has been essentially completely answered in the literature on critical phenomena while the first still poses some interesting problems. In these Lecture Notes we address ourselves exclusively to the second question by considering a model (the Hierarchical Model) in which the first problem is completely answered by construction. This approach is then sufficiently modest to allow for a complete mathematical understanding of the following main problems of RG theory which are : The existence of non-trivial fixed points, their ϵ -expansion, local flows and crossover phenomena and the physical information which can be extracted from these things.

These mathematical problems have been first solved by Eleher and Sinai and most of the proofs can be found in the references by these authors. The present Lecture Notes report these ideas in our realiza-

tion with proofs which differ sometimes essentially from those of Bleher and Sinai. The study of the ϵ -expansion follows our own earlier work, while the existence proof given here is new and our crossover proofs are more detailed than those of Bleher and Sinai.

These Lecture Notes are written in two parts which are distinct in style. In Part I we develop the different aspects of the renormalization group for the Hierarchical Model. These aspects are mostly given in the form of a more intuitive exposition followed by a precise mathematical statement. Those calculations which seem instructive are given in Part I but only the strategy of the proofs is outlined. Our approach to the subject is not along the conventional line because it is exclusively based on statistical mechanics, i.e. thermodynamic quantities appear as derived objects. It may be useful to read one of the review articles by Ma[1], Wilson-Kogut [2] or Fisher [3], to make contact with the more thermodynamic approach.

Part II serves a different purpose : It is a complete mathematical description of all steps used in the arguments of Part I. Many of the results were shown before by Bleher and Sinai and are scattered in the literature. Our proofs are however new and many of them appear here for the first time. The language is that of mathematics and we address readers familiar with functional analysis.

REFERENCES

- [1] Sh. K. MA Introduction to the renormalization group.
Rev. Mod. Phys. 45, 589 (1973).
- [2] K.G. WILSON, I. KOGUT The renormalization group. Phys. Rep.
12C, 75 (1974).
- [3] M.E. FISHER The renormalization group in the theory of critical
behaviour. Rev. Mod. Phys. 46, 597 (1974).
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PART I. HEURISTICS

1. Probabilistic Formulation of the Problem

The success of the RG method rests in part on the fact that statements are only made about a very restricted number of observables of a system. Most of these observables describe the collective behaviour of many degrees of freedom. Typical such observables describe the value of the mean spin of a system, or the fluctuations of this mean.

Probability theory asks similar questions : Given random variables s_1, \dots, s_n , with probability densities $\rho_i(s_i) = \rho(s_i)$, we may ask for the probability density of the sum (or the mean) of the s_i . The answer is well known ; the probability density P_N for $S = s_1 + \dots + s_N$ is for independent random variables,

$$P_N(S) = \int ds_1 \dots ds_N \rho(s_1) \dots \rho(s_N) \delta(S - s_1 - \dots - s_N). \quad (1.1)$$

How does P_N behave in the limit of large N ? The central limit theorem answers this question.

THEOREM 1.1.* Let $\mu = \int s\rho(s) ds < \infty$, $\sigma^2 = \int (s - \mu)^2 \rho(s) ds < \infty$.

Then

$$\lim_{N \rightarrow \infty} P_N(N^{\frac{1}{2}}(S + N\mu)) = (2\pi\sigma^2)^{-\frac{1}{2}} e^{-S^2/2\sigma^2}, \quad (1.2)$$

where the convergence is in the weak sense (i.e.

$$\int f(S) P_N(N^{\frac{1}{2}}(S + N\mu)) dS \rightarrow \int (2\pi\sigma^2)^{-\frac{1}{2}} f(s) e^{-s^2/2\sigma^2} ds$$

for $f \in L_1(dS)$).

* The notation A/BC means always A/(BC) .

The formulae (1.1), (1.2) exhibit many typical features of RG theory : indeed, a question is asked about a sum of variables (sum of spins). This sum is rescaled (renormalized) through $(s - N\mu)/N^{\frac{1}{2}}$. Most importantly the limit has a behaviour which is independent of the details of ρ , and the analogous feature in RG theory is called universality.

The situation described in (1.1), (1.2) corresponds to a free classical discrete spin system with continuous spin (a value in \mathbb{R}), and this can be seen as follows. Choose $\beta > 0$ (the inverse temperature) and set $H(s) = -(\log \rho(s))/\beta$, $H_N(s_1, \dots, s_N) = \sum_{j=1}^N H(s_j)$. $H(s_j)$ is the "energy" of the spin s_j . The expectation for the fluctuation of the sum of the spins $S_N = \sum_{j=1}^N s_j$ is then given by

$$\chi_N^2 = \langle (S_N - \langle S_N \rangle)^2 \rangle / N, \quad (1.3)$$

where

$$\langle f(s_1, \dots, s_N) \rangle = \frac{\int ds_1 \dots \int ds_N e^{-\beta H_N(s_1, \dots, s_N)} f(s_1, \dots, s_N)}{\int ds_1 \dots \int ds_N e^{-\beta H_N(s_1, \dots, s_N)}} \quad (1.4)$$

is the expectation of f in the Gibbs ensemble of statistical mechanics.

Theorem 1.1. implies by inspection that as $N \rightarrow \infty$,

$$\chi_N^2 \rightarrow \sigma^2. \quad (1.5)$$

Thus the fluctuations in any free spin system for which a single spin has finite mean μ and variance σ behave asymptotically like $N^{\frac{1}{2}}\sigma$ as a function of the number N of particles.

In the course of the study of the model, we shall not only concentrate on fixed points but also on the "flow" around them, i.e. on the approach to the fixed points. In fact, from a physical point of view, the latter problem is more important than the former, because it allows to make statements about large but finite systems.

As in probability theory, one can ask which distributions ϕ can occur as limits of initial distributions under some transformations. This is a deep problem, which is completely solved in the case of independent random variables. Also the domain of attraction (= universality class) of each possible limit distribution (which are called the stable distributions in the mathematics literature) is known in this case ; i.e. one can say which distributions "converge" to which limits. Some attempts to make progress in this difficult problem for dependent variables have been made by Sinai[8] and Bleher but they have not yet gone beyond some beautiful but straightforward generalization of phenomena which will show up already in the study of the special case of the hierarchical model. However, the benefit of the probabilistic description of the RG has certainly been to put the notion of universality classes into precise language.

Remarks on Section 1 :

The probabilistic interpretation of the RG has been stressed especially by Jona-Lasinio. Earlier allusions are made in passing in Bleher-Sinai, Baker.

- [4] G. JONA-LASINIO : The renormalization group : A probabilistic view. *Il Nuovo Cimento* 26B, 99 (1975).
- [5] P.M. BLEHER, Ja.G. SINAI : Investigation of the critical point in models of the type of Dyson's Hierarchical Model. *Commun.Math. Phys.* 33, 23 (1973).
- [6] G.A. BAKER : Ising model with a scaling interaction. *Phys. Rev. B* 5, 2622 (1972).

A detailed study of sequences of independent random variables can be found in

- [7] B.V. GNEDENKO, A.N. KOLMOGOROV : Limit distributions for sums of independent random variables. Cambridge Mass. 1954, Addison Wesley.
- [8] Ja.G. SINAI : Self-similar probability distributions. *Theory of probability and its applications* 21, 64 (1976).

2. The RG-Transformation for the Hierarchical Model

We start this section by defining the model. The Hierarchical Model is a model of continuous spins on a one-dimensional lattice. If the lattice has N points, the spins will be called s_1, \dots, s_N . For every real function f and every $N = 2^M$ we define the Hamiltonian $\mathcal{H}_{N,f}$ of the system to be

$$\mathcal{H}_{N,f} = \mathcal{H}_N + \sum_{j=1}^N f(s_j), \quad (2.1)$$

$$\mathcal{H}_N = - \sum_{k=1}^M \frac{c^k}{2^{2k+1}} \sum_{j=0}^{2^{M-k}-1} \left(\sum_{l=1}^{2^k} s_{j2^{k+1}+l} \right)^2. \quad (2.2)$$

The constant c is real and $1 < c < 2$.

We do not discuss at this point for which values of c and for which choices of f the Hamiltonian actually defines a thermodynamically stable system. Let us now describe the heuristics of Eq. (2.2). The Hamiltonian \mathcal{H}_N is the sum of terms on "levels" $k=1, \dots, M$. On each level k , the 2^M spins are grouped into disjoint blocks of 2^k spins each and the interaction for such a block is then

$$- \frac{c^k}{2^{2k+1}} \left(\sum_{l=1}^{2^k} s_{j2^{k+1}+l} \right)^2.$$

This is usually visualized graphically as follows:

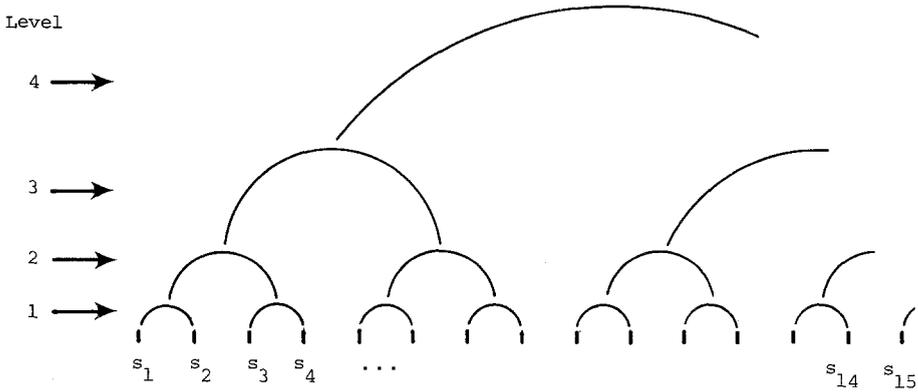


Figure 1. The hierarchical structure of the interaction

Let us study the interaction between s_i and s_j , $i \neq j$. By the nature of the Hamiltonian, there will be a lowest level for which s_i and s_j lie in the same block, say the level k . Then the interaction between s_i and s_j is $-(c/4)^k$. On the other hand, the fact that the lowest level is k implies $|i-j| \geq 1$ and $|i-j| \leq 2^k - 1$. It is thus reasonable to say that the interaction potential is about of the form $|i-j|^{\log_2(c/4)}$ but this is to be taken with a grain of salt because the model is not translation invariant. We thus see that the range of the interaction depends on c .

Most often RG theory is done in varying dimension for short range interactions. In the case of the Hierarchical Model, the situation is reversed in that the dimension is fixed and the range of the interaction is varied. While this is unusual, it has the advantage

of being more easily implementable from a mathematical point of view than the notion of fractional space dimension.

Let us now assume $f(s)$ is sufficiently increasing at infinity so that $\int_{i=1}^N ds_i s_i^k \exp(-\beta \mathcal{H}_{N,f}(s))$ exists. Then the model is defined for all finite volumes and we may discuss its partition function. In particular, we shall consider the probability density for the sum of spins, as in Section 1, at inverse temperature $\beta > 0$. It is

$$P_{N,f}^{(\beta)}(S) = \frac{\int ds_1 \dots ds_N \delta(S - s_1 - \dots - s_N) e^{-\beta \mathcal{H}_{N,f}}}{\int ds_1 \dots ds_N e^{-\beta \mathcal{H}_{N,f}}} \quad (2.3)$$

We shall now compare $P_{2N,f}^{(\beta)}$ and $P_{N,g}^{(\beta)}$, using the explicit definition (2.3) and the special form of the Hamiltonian (2.2). Observe that for $k \geq 1$,

$$\sum_{l=1}^{2^k} s_{j2^{k+1}} = \sum_{l=1}^{2^{k-1}} s_{j2^{k+2} \cdot l-1} + s_{j2^{k+2} \cdot l}$$

Therefore, for $N = 2^M$, $M \geq 0$, we find

$$\begin{aligned} & \mathcal{H}_{2N}(s_1, \dots, s_{2N}) \\ &= - \sum_{k=1}^{M+1} \frac{c^{k-1}}{2^{2(k-1)+1}} \sum_{j=0}^{2^{M+1-k}-1} \left(\sum_{l=1}^{2^{k-1}} \frac{c^{\frac{1}{2}}}{2} (s_{j2^{k+2} \cdot l-1} + s_{j2^{k+2} \cdot l}) \right)^2 \\ &= \mathcal{H}_N((s_1+s_2) c^{\frac{1}{2}}/2, (s_3+s_4) c^{\frac{1}{2}}/2, \dots, (s_{2N-1} + s_{2N}) c^{\frac{1}{2}}/2) \\ &- \frac{c}{8} \sum_{j=0}^{2^M-1} (s_{2j+1} + s_{2j+2})^2 \end{aligned}$$

Therefore we find, for any measurable function F ,

$$\begin{aligned} & \int ds_1 \dots ds_{2N} F \left(\sum_{j=1}^{2N} s_j \right) \exp(-\beta \mathcal{K}_{2N,f}(s_1, \dots, s_{2N})) \\ = & \int ds_1 \dots ds_{2N} \prod_{j=1}^{2N} \pi \exp(-\beta f(s_j)) \prod_{j=1}^N \exp(\beta c(s_{2j-1} + s_{2j})^2/8) \\ & \cdot F \left(\sum_{j=1}^{2N} s_j \right) \exp(-\beta \mathcal{K}_N((s_1 + s_2)c^{\frac{1}{2}}/2, \dots, (s_{2N-1} + s_{2N})c^{\frac{1}{2}}/2)), \end{aligned}$$

which, upon setting

$$t_j = (s_{2j-1} + s_{2j})c^{\frac{1}{2}}/2, \quad u_j = (s_{2j-1} - s_{2j})/2,$$

becomes

$$\begin{aligned} & (2/c^{\frac{1}{2}})^N \int dt_1 \dots dt_N F(2c^{-\frac{1}{2}} \sum_{j=1}^N t_j) \exp(-\beta \mathcal{K}_N(t_1, \dots, t_N)) \\ & \cdot \prod_{j=1}^N \pi \int du_j \exp(-\beta f(t_j c^{-\frac{1}{2}} + u_j) - \beta f(t_j c^{-\frac{1}{2}} - u_j)) \exp(\beta t_j^2/2) \\ = & \int dt_1 \dots dt_N F(2c^{-\frac{1}{2}} \sum_{j=1}^N t_j) \exp(-\beta \mathcal{K}_{N,g}(t_1, \dots, t_N)), \end{aligned}$$

where g is defined by

$$e^{-\beta g(t)} = e^{\beta t^2/2} (2/c^{\frac{1}{2}}) \int du e^{-\beta f(tc^{-\frac{1}{2}} + u) - \beta f(tc^{-\frac{1}{2}} - u)}. \quad (2.4)$$

The Equation (2.4) defines a transformation

$$f \longrightarrow g = \mathcal{N}_p^{(\beta)}(f),$$

and upon inserting our calculations into (2.3) we find the important relation

$$P_{2N, f}^{(\beta)}(S) = (c^{\frac{1}{2}}/2) P_{N, \mathcal{N}_P}^{(\beta)}(\beta)_{(f)} \left((c^{\frac{1}{2}}/2) S \right) . \quad (2.4a)$$

What have we now achieved ? We have related the probability densities corresponding to two different numbers of spins (namely N and $2N$) through a change of scale (1 goes to $2/c^{\frac{1}{2}}$) and by a change of Hamiltonian $\mathcal{H}_{., f} \rightarrow \mathcal{H}_{., \mathcal{N}_P}(\beta)_{(f)}$. Putting it slightly differently : A simultaneous change of scale and of the Hamiltonian has no effect. The semigroup formed by these simultaneous transformations is called the renormalization group. The Hierarchical Model is an especially simple system insofar as the change of Hamiltonian concerns only the single spin distribution f . In the general framework of the renormalization group theory the transformation of the Hamiltonian involves other terms, too. The simple structure of the RG transformation for the Hierarchical Model will make a rigorous mathematical discussion possible, while the typical features of RG theory are preserved.

What can these RG equations be used for ? First of all we recast them into a form which shows the similarities with the probabilistic aspects discussed in Section 1. The quantity P_N which we considered there satisfied the equation

$$P_{2N}(S) = \int dT P_N(S/2-T) P_N(S/2+T),$$

and the central limit theorem (Theorem 1.1) asserted (in the case of zero mean $\mu = 0$)

$$\lim_{M \rightarrow \infty} P_{2^M} (2^{M/2} S) \longrightarrow \text{Gaussian} . \quad (2.5)$$

For $N = 2^M$ we may also decompose the Hamiltonian as the following sum :

$$\mathcal{H}_{2N,f}(s_1, \dots, s_{2N}) = \mathcal{H}_{N,f}(s_1, \dots, s_N) + \mathcal{H}_{N,f}(s_{N+1}, \dots, s_{2N}) - \frac{1}{2} c^{M+1} 2^{-2M-2} \left(\sum_{j=1}^{2N} s_j \right)^2.$$

Then by a sequence of transformations similar to those leading to (2.4) we get

$$P_{2N,f}^{(\beta)}(S) = \text{const.} \exp(\beta c^{M+1} 2^{-2M-2} S^2/2) \cdot \int dT P_{N,f}^{(\beta)}(S/2 - T) P_{N,f}^{(\beta)}(S/2 + T).$$

In analogy with (2.5) we may consider

$$P_{2^M,f}^{(\beta)}\left(\left(2/c^{1/2}\right)^M S\right) = \mathcal{R}_{2^M,f}^{(\beta)}(S),$$

which then satisfies

$$\mathcal{R}_{2^{M+1},f}^{(\beta)}(S) = \text{const.} \exp(\beta S^2/2) \cdot \int du \mathcal{R}_{2^M,f}^{(\beta)}\left(S c^{-1/2} + u\right) \mathcal{R}_{2^M,f}^{(\beta)}\left(S c^{-1/2} - u\right), \quad (2.6)$$

as compared to $\mathcal{R}_{2^M}(S) = P_{2^M}(2^{M/2} S)$, in the case discussed in Section 1 which satisfies

$$\mathcal{R}_{2^{M+1}}(S) = \text{const.} \int du \mathcal{R}_{2^M}(S 2^{-1/2} + u) \mathcal{R}_{2^M}(S 2^{-1/2} - u). \quad (2.7)$$

The equation (2.6) is very similar to (2.7) which we discussed in Section 1. But the very regular situation described in Equ. (2.5) may

now change drastically for one value of β , called the inverse critical temperature. Then the fluctuations can be for example of order N^τ , $\tau \neq 1$, and S could tend to Gaussian distribution with variance $N^{\tau/2} \sigma$;

$$\chi_N^2 / N^{\tau-1} \rightarrow \sigma^2, \quad \tau \neq 1. \quad (2.8)$$

(In our case $\tau = 2 - \log_2 c$).

Finally, there is the possibility that $\tau \neq 1$ and in addition $S_N / N^{\tau/2}$ does not tend to a Gaussian distribution, but to some other distribution Φ . This third case

$$P_{N,f}^{(\beta)} \left(N^{\tau/2} (S + N \mu) \right) \rightarrow \Phi(S) \neq (2\pi\sigma^2)^{-\frac{1}{2}} e^{-S^2/2\sigma^2}, \quad (2.9)$$

is the most interesting one from a physical point of view, and the limit Φ is called a nontrivial critical spin distribution; or (the exponential of) a critical Hamiltonian. We prefer the first interpretation, and this is the reason for having exposed the RG in the probabilistic framework . (In mathematics Φ would correspond to the distribution of a sum of dependent random variables.) We shall see that in the Hierarchical Model behaviour of the type Eq.(2.9) occurs. The purpose of these Lecture Notes is among others to study this generalized form of a central limit theorem for the Equation (2.6). But we view the limit itself as a fixed point of the transformation

$\mathcal{R}_{N,f}^{(\beta)} \rightarrow \mathcal{R}_{N+1,f}^{(\beta)}$ defined by Equation (2.6). In fact, we shall not work with (2.6), which we used to show the connection between the RG theory and the central limit theorem, but we shall rather concentrate on the transformation $\mathcal{N}_P^{(\beta)}$ defined in (2.4), which also describes the scaling behaviour of the main object, namely $P_{N,f}^{(\beta)}$, which is defined in Eq. (2.3).

These Lecture Notes are then a study of the transformation $\mathcal{N}_P^{(\beta)}$.

Two main methods for this study are used :

- M 1) Look for a fixed point of the map $\mathcal{N}_P^{(\beta)}$. Then under suitable conditions, the behaviour of the map in a neighborhood of the fixed point is completely described in terms of the tangent map at the fixed point. We shall see later that $\mathcal{N}_P^{(\beta)}$ has fixed points which are not Gaussian, and these are the ones of special interest to us.
- M 2) Follow trajectories globally. This method is much less systematic than the first one and our results are maybe mathematically not so appealing.

The above methods allow both for strong results about the system. From a physical point of view the results provided through M 1 and M 2 are distinct.

M 1 allows to determine the critical indices, i.e. to determine the behaviour of thermodynamic variables when the temperature reaches the critical temperature. M 1 corresponds to the so-called scaling limit. The fact that the result is independent of some class of functions f reflects what is called the universality character of the RG method.

M 2 allows to prove, for suitable functions f in $\mathcal{K}_{N,f}$, and for suitable observables, the existence of the thermodynamic limit, i.e. the limit $M \rightarrow \infty$ in (2.6), at temperatures near, but not equal to a specific temperature, called the critical temperature. In addition it implies that the mean spin and the correlation length are finite when the temperature is not critical. Finally, the existence of a phase transition at the critical temperature follows. (Such results can often be obtained by totally different arguments, but the RG treatment

seems particularly nice in the context of the Hierarchical Model. Furthermore the results on finite correlation length outside the critical temperature are not known except for the Ising model).

As we have seen above, the Hierarchical Model has the property that its RG transformation $\mathcal{N}_P^{(\beta)}$ is a known transformation on the space of single spin distribution. This is not the case for a general model, but believed to be approximately true for large N . Whenever this should be the case for a transformation sufficiently similar to $\mathcal{N}_P^{(\beta)}$ (e.g. convolution of several factors and a Gaussian factor) the ideas of these Lecture Notes could be carried over. However, the determination of a "correct" approximate RG transformation is a very hard problem for a general microscopic Hamiltonian, and we do not pursue this question any further.

In the next section, we shall discuss the existence of a nontrivial fixed point of the transformation $\mathcal{N}_P^{(\beta)}$, and we shall come back to the application of Method M 1 in later sections.

Remarks on Section 2 :

The Hierarchical Model has been invented by Dyson to show that one-dimensional systems may exhibit phase transitions if they have long-range forces.

- [9] F.J.DYSON : Existence of a phase-transition in a one-dimensional Ising ferromagnet . Commun. Math.Phys. 12, 91 (1969).
- [10] F.J.DYSON : An Ising ferromagnet with discontinuous long-range order. Commun. Math. Phys. 21, 269 (1971).

Baker reinvented the model and pointed out that the RG acted on the single spin distribution. He also calculated critical indices.

- [11] G.A. BAKER, Jr : Ising model with a scaling interaction. Phys. Rev. B5 , 2622, (1972).
- [12] G.A. BAKER, Jr ; G.R. GOLNER : Spin-spin correlations in an Ising model for which scaling is exact. Phys. Rev. Lett. 31, 22 (1973).
- [13] G.A. BAKER, Jr, S. KRINSKY : Renormalization group structure for translationally invariant ferromagnets. Journ. math. Phys. 18, 590 (1977).

The first rigorous mathematical work was done in the paper by Bleher and Sinai [5], on the case of a Gaussian fixedpoint (with $2^{\frac{1}{2}} < c < 2$).